ZW

stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE ZW 48/75 JUNE

D. LEIVANT

FAILURE OF COMPLETENESS PROPERTIES OF INTUITIONISTIC PREDICATE LOGIC FOR CONSTRUCTIVE MODELS

Preliminary Report

ZW

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 02D99, 02E05, 02G20
02F25, 02H25

Failure of completeness properties of intuitionistic predicate logic for constructive models

by

D. Leivant

ABSTRACT

We consider a principle of constructivity RED which states that every decidable predicate over the natural numbers is (weakly) recursively enumerable (r.e.). RED is easily seen to be derived from Church's thesis CT_0 ("every construction is given by a recursive function"). Results:

(1) RED implies that the species of valid first order predicate schemata is not r.e., and hence - that intuitionistic first order predicate logic L_1 is incomplete.
(2) We construct a specific schema of L_1 which is valid if RED, but unprovable in L_1.
(3) The two results above hold even when validity is generalized to validity with KREISEL-TROELSTRA [70]'s choice sequences as parameters.
(4) The method is used also to construct a schema of L_1, unprovable in L_1, but of whose all metasubstitutions with Σ_1^0 number theoretic predicates are provable in Heyting's arithmetic A. This is a simple bound on possible improvements of the absoluteness result of LEIVANT [75].

KEY WORDS & PHRASES: *Intuitionistic predicate logic, completeness, Church's thesis, absoluteness.*

1. INTRODUCTION

1.1. HISTORICAL NOTE

Gödel was the first one to note that (weak) completeness of L_1 implies principles which are dubious from an intuitionistic viewpoint, namely - Markov's schema with choice parameters. Gödel's argument is presented in KREISEL [62] §6.

According to this, in order to show that

(*) *Church's thesis* CT_0 *implies the (weak) incompleteness of* L_1

it suffices to prove the inconsistency of CT_0 with the version of Markov's schema mentioned above (although CT_0 is consistent with Markov's schema without function parameters). This step was performed only recently by TROELSTRA [74].

Instead, KREISEL sketched in [70] p.133 a direct proof of (*) using a recursion theoretic argument. Kreisel's sketch was elaborated (independently) in VAN DALEN [72] §4 and in LEIVANT [72] §4, both using a modified recursion theoretic argument which utilizes a lemma due to C. Jockusch.

The present note is a simplification of Kreisel's method: both the recursion theoretic techniques(*) and the metamathematical principles used are reduced. The application mentioned under (4) in the abstract is new.

1.2. THE IDEA OF THE PROOF (OF (1))

We first note that the theory of primitive recursive computations can be mimicked in the language of L_1 (2.1 below). That is, for every primitive recursive (p.r.) function f there is a schema $G[Eq,Z,S,F_1,\ldots,F_n]$ of L_1, s.t. (intuitively) every model of G contains a submodel isomorphic to ω, where Eq, Z and S are interpreted as =,0 and the successor relation $y = x^+$ resp., and where $F_1,\ldots,F_n$ mimick the computation instructions for f, so

---
(*) Prof. Troelstra has noted that the recursion theoretic idea here is close to VAUGHT [60].

that $F_i(x_1,\ldots,x_r,y)$ corresponds to $f_i(x_1,\ldots,x_r) = y$, and $f_n = f$.

Hence, if R(e) is a certain arithmetical property of r.e. sets W_e, we have a schema R_e^* of L_1 for each $e < \omega$ s.t.

(**) $\quad R_e^*$ *is true in every structure* $\Rightarrow$ R(e) *holds in* ω.

If, in addition, R_e^* is of the form $U \to E$, where U is purely universal and E is purely existential, then R_e^* is persistent under extension of models, and consequently the converse of (**) holds as well.

Thus, for a numeric predicate R for which all R_e^*'s are of the form above,

(***) $\quad \{ e \mid R_e^* \textit{ is valid} \} = \{ e \mid R(e) \}.$

I.e., an r.e. set of schemata of L_1 corresponds to an index set of r.e. sets. By the Rice-Shapiro theorem, an r.e. index set must be characterized by a particularly simple condition; this agrees with (***) because each R_e^* is indeed of a very simple form.

Let us now force certain (unary) predicate letters P,Q,... occurring in $R_e^* \equiv U \to E$ to be decidable, i.e. - we take in place of R_e^* the schema

$$U \;\&\; \forall x\,(\,Px \vee \neg Px\,) \;\&\ldots \;\to\; E$$

which is still of the form $U' \to E$ with U' purely universal. But now, if RED is assumed, the predicates P,Q,... must range over r.e. sets in every model isomorphic to $\langle \omega,=,0,\lambda x.x^{+}\rangle$. Each R_e^* interprets then a number theoretic statement of the form $\forall x A(e,W_x)$. Roughly speaking, the implicit quantification over species involved in stating "R_e^* is valid" corresponds, under RED, to an additional *universal numeric* quantifier in R(e).

We may now pick up a predicate R for which $\{ e \mid R(e) \}$ is (provably) not r.e., and by (***) the species

$$S_1 \;:\equiv\; \{ \ulcorner R_e^* \urcorner \mid R_e^* \textit{ is valid} \}$$

is also not r.e. But $S_2 :\equiv \{\ulcorner R_e^* \urcorner\}_{e<\omega}$ is recursive, since this is just a species of syntactically characterized schemata. Consequently the species

$$S_3 \;:\equiv\; \{ \ulcorner F \urcorner \mid F \textit{ is a valid schema of } L_1 \}$$

is not r.e., because $S_1 = S_2 \cap S_3$.

1.3. - 1.5. FORMAL SETTING

1.3. LANGUAGE

Our metamathematical setting is the intuitionistic theory of species L_2. An inspection on the proofs below shows that actually only a small fragment of L_2 is used, but since it is of little value for our purpose to delimit this fragment precisely we do not bother to do so.

PRAWITZ ([65] p.72) has shown that Heyting's arithmetic A is interpretable in L_2; however, to avoid notational complications we shall consequently use explicitly low case m,n,p etc. for variables ranging over numbers (i.e. over the definable species of natural numbers), and low case x,y,z etc. for unrestricted first order variables. We also assume that A as well as L_2 contain symbols and defining equations for all p.r. functions.

The system we study is first order intuitionistic predicate logic L_1, without equality and without function symbols. We shall find it convenient to refer also to L_1 extended to the language with all constants of A (=, numerals and p.r. functions). We write L_1A for the resulting system.

1.4. THE VALIDITY PREDICATE

Tarski's definition of a Π^1_1 validity predicate Val for L_1 (analogously to the classical case) is standard. Semi-formal details can be found e.g. in MENDELSON [64] pp.50-53. The central property of Val is of course

$$(1) \qquad \vdash_{L_2} \underline{\mathrm{Val}}(\ulcorner G \urcorner) \;\leftrightarrow\; \forall D^1 \,\forall \vec{X}\; G^D[\vec{X}/\vec{P}]$$

for each schema G of L_2, where $\vec{P}$ is the list of all predicate parameters occurring in G, $\vec{X}$ is a corresponding list of predicate variables, D^1 is a (unary) predicate variable and G^D comes from G by restricting all quantifiers to D.

To avoid confusion one should note that the predicate <u>Val</u> does not express any analysis of the notion of constructive truth (contrary to truth definitions like Gödel's functional interpretation, Beth's and Kripke's semantics and the various notions of realizability). Here the constructive content of L_1 shows only through the constructive meaning given (in L_2) to the logical connectors occurring in <u>Val</u>. Since we certainly intend to give to the first order connectors the same meaning in L_2 as in L_1, our discussion is insensitive to any specific analysis of contructivity.

1.5. THE SCHEMA RED

RED, for "Recursive Enumerability of Decidable predicates" reads

$$\forall n \, [\, Pn \vee \neg Pn \,] \rightarrow \neg\neg \exists e \forall n \, [\, Pn \leftrightarrow \{e\}n \simeq 0 \,]$$

where {e} is Kleene's notation for the e'th general recursive function, i.e. -

$$\{e\}n \simeq m \;:\equiv\; \exists z \; T(e,n,z) \;\&\; U(z) = m.$$

Let us see how RED compares with the metamathematical principles used by VAN DALEN [72] which, slightly restated to fit into our formalism, read as follows.

(i) The schema of choice AC_{VN} in the form

$$\forall X^2 \, [\, \forall x \exists n \leq 1 \; X(x,n) \rightarrow \exists \alpha^{V \rightarrow N} \forall x \; X(x, \alpha x) \,]$$

where V→N is the type of lawlike functions from the universe of first order objects to natural numbers (p.80, 1.4-5).

(ii) The principle of dependent choice DC_{NV} (79_3 there)

$$\forall X^2 \, [\, \forall n \exists x \; X(n,x) \rightarrow \forall y \exists \alpha^{N \rightarrow V} \, [\, \alpha(0) = y \;\&\; \forall n \; X(\alpha(n), \alpha(n+1)) \,]\,]$$

(iii) Church's thesis CT:

$$\forall\alpha^{N\to N}\exists e\forall n\ \{e\}n \simeq \alpha(n)$$

Note that even the positive version of RED is derivable from

CT_0: $\forall n\exists m\ A(n,m) \to \exists e\forall n\ A(n,\{e\}n)$

which is an immediate consequence of AC_{NN} and CT. To see this, assume

$$\forall n\ [\ Pn \vee \neg Pn\ ];$$

then

$$\forall n\exists m\ [\ (\ m = 0 \to Pn\ )\ \&\ (\ m \neq 0 \to \neg Pn\ )\ ]$$

which implies by CT_0

$$\exists e\forall n\ [\ \{e\}n \simeq 0 \leftrightarrow Pn\ ],$$

i.e. - RED. Actually the last step requires only

$CT_0!$: $\forall x\exists! y\ A(x,y) \to \exists e\forall x\ A(x,\{e\}x)$.

$CT_0!$ was shown by LIFSHITS [74] to be strictly weaker (in A) than CT_0.

2. CHARACTERIZATION OF (WEAKLY) RECURSIVELY INSEPARABLE PAIRS OF R.E. SETS IN L_1 VIA RED

2.1. A FINITE AXIOMATIZATION OF PRIM. REC. COMPUTATIONS

In the language of L_1, fix three predicates Eq(x,y), Z(x) and S(x,y). We think of these as representing equality, zero and the successor relation. Let A^S (≡ the axiom of the theory of successor) be the conjunction of the closure of the following formulae of L_1.

(1) $Eq(x,x)$

(2) $Eq(x,y)\ \&\ Eq(x,z) \rightarrow Eq(y,z)$

(3) $Z(x) \rightarrow [\ Z(y) \leftrightarrow Eq(x,y)\ ]$

(4) $S(x,y) \rightarrow [\ S(x,z) \leftrightarrow Eq(y,z)\ ]$

(5) $S(x,y) \rightarrow [\ S(z,y) \leftrightarrow Eq(x,z)\ ]$

(6) $Z(x) \rightarrow \neg S(y,x)$

(7) $\exists z\ Z(z)$

(8) $\exists y\ S(x,y)$

Let $\{f_n\}_{n<\omega}$ be an enumeration of all p.r. functions where each f_n is defined in terms of $\{f_i\}_{i<n}$ through the following familiar schemata.

(1) (zero) $f_n(x) = 0$

(2) (successor) $f_n(x) = x^+$ (:= the successor of x)

$(3)_{q,i}$ (projection) $f_n(x_0,\ldots,x_q) = x_i$ $(i \leq q)$

$(4)_{q,r}$ (composition) $f_n(x_0,\ldots,x_q) = f_m(f_{i_0}(x_0,\ldots,x_q),\ldots,f_{i_r}(x_0,\ldots,x_q))$ $(m,i_0,\ldots,i_r<n)$

(5) (recursion) $f_n(0,x) = f_m(x)$

$f_n(y^+,x) = f_\ell(y,x,f_n(y,x))$ $(m,\ell<n)$

These equations may be axiomatized by

(1^*) $F_n(x,v) \leftrightarrow Z(v)$

(2^*) $F_n(x,v) \leftrightarrow S(x,v)$

$(3^*)_{q,i}$ $F_n(x_0,\ldots,x_q,x_i)$ $(i \leq q)$

$(4^*)_{q,r}$ $\quad \bigwedge_{j\le r} F_{i_j}(x_0,\dots,x_q,y_j) \rightarrow [\ F_n(x_0,\dots,x_q,v) \leftrightarrow F_m(y_0,\dots,y_r,v)\ ]$

(5^*) $\quad Z(z) \rightarrow [\ F_n(z,x,v) \leftrightarrow F_m(x,v)\ ]$.&.

$S(y,w)\ \&\ F_n(y,x,z) \rightarrow [\ F_n(w,x,v) \leftrightarrow F_\ell(y,x,z,v)\ ]$

For every n, let C_n be the conjunction of the closure of all formulae corresponding to the defining equations for f_i, $i \le n$, and

$$A_n \ :\equiv\ A^S \ \&\ C_n$$

2.2. We refer below (2.6) to schemata $G_{m,n}$ of the form

$$D[P,Q] \rightarrow \neg\neg E[F_m,F_n;P,Q]$$

where P,Q are unary and F_m,F_n are binary predicate letters (m<n), and where

$$D[P,Q] \ :\equiv\ \forall x\ (\ Px \ \vee\ \ Px\ )\ \&\ \forall x\ (\ Qx \ \vee\ \neg Qx\ ).$$

E is here a fixed schema of L_1, free of $\rightarrow$ and $\forall$, where only P and Q occur negated.

Given a schema G of L_1, let us write G^ω for the schema of L_1A which comes from G by replacing the predicates Eq(x,y), Z(x), S(x,y),..., $F_i(x_1,\dots,x_n,y)$,... by $x = y$, $x = 0$, $y = x^+$, ..., $y = f_i(x_1,\dots,x_n)$,... (and replacing the "general" variables of L_2, $x_0,x_1,\dots$ say, by corresponding numeric variables: $n_0,n_1,\dots$).

We have now

(1) $$\vdash_{L_2} \underline{\mathrm{Val}}(\ulcorner G_{m,n}{}^{\omega}\urcorner) \leftrightarrow \forall P,Q\ \{\ D[P,Q] \rightarrow \neg\neg E[F^\omega_m,F^\omega_n;P,Q]\ \}$$

(Recall that $F^\omega_i(x,y)\ :\equiv\ y = f_i(x)$.)

We also write

(2) $$\underline{\mathrm{Val}}^{\Sigma^0_1}(\ulcorner G_{m,n}{}^{\omega}\urcorner) \ :\equiv\ \forall e_1,e_2\ \{\ D[W_{e_1},W_{e_2}] \rightarrow \neg\neg E[W_{e_1},W_{e_2},F^\omega_m,F^\omega_n]\ \}$$

where

$$W_e(x) \;:\equiv\; \{e\}x \simeq 0$$

$$\equiv\; \exists z\; [\; T(e,x,z) \;\&\; Uz = 0 \;]$$

Note that $\underline{Val}^{\Sigma_1^0}(\; G^{\omega}_{m,n}\;)$ is a purely arithmetical formula.

2.3. LEMMA.

$$\vdash_{L_2} f_n(\bar{p}_0,\ldots,\bar{p}_q) = \bar{s} \;\rightarrow$$

$$[\; A_n \;\&\; Z(y_0) \;\&\; \bigwedge_{i<\underline{max}[p_0,\ldots,p_q,s]} S(y_i,y_{i+1}) \;\rightarrow\; F_n(y_{p_0},\ldots,y_{p_q},y_s)\;]$$

PROOF: By primary induction on n and secondary induction on $\underline{max}[p_0,\ldots,p_q]$. If f_n is one of the three basic functions (zero, successor and projection) the statement is trivial. If f_n is defined by composition,

$$f_n(\bar{p}_0,\ldots,\bar{p}_q) = f_m(f_{i_0}(\bar{p}_0,\ldots,\bar{p}_q),\ldots,f_{i_r}(\bar{p}_0,\ldots,\bar{p}_q)),$$

then for the equations

$$(1)\quad \begin{cases} f_{i_0}(\bar{p}_0,\ldots,\bar{p}_q) = \bar{t}_0 \\ \quad\vdots \\ f_{i_r}(\bar{p}_0,\ldots,\bar{p}_q) = \bar{t}_r \\ f_m(\bar{t}_0,\ldots,\bar{t}_r) = \bar{s} \end{cases}$$

we have by induction hypothesis

$$(2)\qquad A_j \;\&\; Z(y_0) \;\&\; \bigwedge_{i<n_j} S(y_i,y_{i+1}) \;\rightarrow\; F_j(y_{k_0},\ldots,y_{k_j})$$

for the r + 2 values of $\langle j,n_j,k_0,\ldots,k_j\rangle$ corresponding to the equations (1). Using schema (4^*) which defines F_n here we get

$$(3)\qquad A_n \;\&\; Z(y_0) \;\&\; \bigwedge_{i<M} S(y_i,y_{i+1}) \;\rightarrow\; F_n(y_{p_0},\ldots,y_{p_q},y_s)$$

where

$$M := \underline{\max}[n_{i_0},\dots,n_{i_r},n_m] \geq n$$

(note that $i_0,\dots,i_r,m<n$). Since, however, A^S implies

$$\bigwedge_{\underline{\max}[p_0,\dots,p_q,s]\leq i<M} \exists y_i \; S(y_i,y_{i+1})$$

we can cut down the length of the premise of (3) and obtain

$$A_n \quad \& \quad Z(y_0) \quad \& \quad \bigwedge_{i\leq\underline{\max}[p_0,\dots,p_q,s]} S(y_i,y_{i+1}) \rightarrow F_n(y_{p_0},\dots,y_{p_q},y_s)$$

as needed.

The case that f_n is defined by recursion is treated similiarly, except that here we proceed also through the secondary induction. □

2.4. PROPOSITION:

$$\vdash_{L_2} \underline{Val}(\ulcorner A_n \rightarrow G_{m,n}\urcorner) \leftrightarrow \underline{Val}(\ulcorner G_{m,n}^{\omega}\urcorner)$$

PROOF

(i) Assume $\underline{Val}(\ulcorner A_n \rightarrow G_{m,n}\urcorner)$. Then trivially (by second-order ∀-elimination) $\underline{Val}(\ulcorner A_n^{\omega} \rightarrow G_{m,n}^{\omega}\urcorner)$. But A_n^{ω} is true, since the arithmetical $=,0,x^+,f_0,\dots,f_n$ satisfy A_n. So $\underline{Val}(\ulcorner G_{m,n}^{\omega}\urcorner)$.

(ii) Assume $\underline{Val}(\ulcorner G_{m,n}^{\omega}\urcorner)$, and fix P,Q, i.e.-

$$(1) \qquad D^{\omega}[P,Q] \rightarrow \neg\neg E^{\omega}[\lambda pq.f_m(p)=q,\lambda pq.f_n(p)=q;P,Q]$$

Towards proving $A_n \rightarrow G_{m,n}[P,Q]$ assume A_n and $D[P,Q]$. In particular then $D^{\omega}[P,Q]$, and so by (1) $\neg\neg E^{\omega}$.

We proceed by induction on the complexity of E, making essential use of the absence of → and ∀. This we do by looking at closed subformulae $H^{\omega}(\vec{n})$ of E^{ω} and proving

$$(2) \qquad H^{\omega}(\vec{n}) \rightarrow [\; A_n \rightarrow \exists\vec{x}\; H(\vec{x}) \;]$$

Basis. If H is P,Q,¬P or ¬Q, then (2) is trivial. If H is F_m or F_n, $H^\omega \equiv f_n(p) = q$ say, then by 2.3

$$(3) \quad A_n \;\&\; Z(y_0) \;\&\; \bigwedge_{i<\underline{\max}[p,q]} S(y_i,y_{i+1}) \rightarrow F_n(y_p,y_q)$$

and by A^S (which is one of the conjuncts of A_n) we can cut down the premiss of (3) and obtain

$$A_n \rightarrow \exists x_1,x_2\, F_n(x_1,x_2)$$

Induction step. If $H \equiv H_1 \;\&\; H_2$ or $H \equiv H_1 \vee H_2$ then (2) for H is implied trivially by (2) for H_1,H_2 (ind. hyp.). If $H^\omega \equiv \exists m\, J^\omega(m,\vec{n})$ then for some m $J(\bar{m},\vec{n})$ and so by ind. hyp.

$$A_n \rightarrow \exists y \exists \vec{x} J(y,\vec{x}).$$

Now setting $H \equiv E$ we get $E^\omega \rightarrow E$ and since we have $\neg\neg E^\omega$, we get $\neg\neg E$ as required. □

2.5. PROPOSITION

$$\vdash_{L_2+\mathrm{RED}} \underline{\mathrm{Val}}(\ulcorner G_{m,n}{}^{\omega}\urcorner) \leftrightarrow \underline{\mathrm{Val}}^{\Sigma_1^0}(\ulcorner G_{m,n}{}^{\omega}\urcorner)$$

PROOF. The implication from left to right is trivial. Assume, on the other hand,

$$(1) \quad \underline{\mathrm{Val}}^{\Sigma_1^0}(\ulcorner G_{m,n}{}^{\omega}\urcorner) \equiv \forall e_1,e_2 \{ \forall n (W_{e_1} n \vee \neg W_{e_1} n) \;\&\; \forall n (W_{e_2} n \vee \neg W_{e_2} n) \rightarrow \neg\neg E^\omega[\lambda pq.f_m(p)=q, \lambda pq.f_n(p)=q;\ W_{e_1}, W_{e_2}] \}$$

Towards proving $\underline{\mathrm{Val}}(\ulcorner G_{m,n}{}^{\omega}\urcorner)$ fix P and Q and assume

$$\forall n (Pn \vee \neg Pn) \;\&\; \forall n (Qn \vee \neg Qn).$$

By RED then

$$\neg\neg\exists e_1, e_2 \; [\; P \equiv W_{e_1} \quad \& \quad Q \equiv W_{e_2} \;].$$

By (1) and predicate logic we thus get the antecedent $\neg\neg E[F_m, F_n;\, P,Q]$ of $G_{m,n}{}^{\omega}[P,Q]$. □

2.6. PROPOSITION

$$\vdash_{L_2+\mathrm{RED}} \underline{\mathrm{Val}}(\ulcorner A_n \to G_{m,n}\urcorner) \leftrightarrow \underline{\mathrm{Val}}^{\Sigma^0_1}(\ulcorner G_{m,n}{}^{\omega}\urcorner)$$

PROOF: Immediate from 2.4 and 2.5. □

Fix now

$$\begin{aligned} E[F_m, F_n;\, P,Q] \;:\equiv\;& \exists x\,[\,\exists y\, F_n(y,x) \;\&\; P(x)\,] \\ \vee\;& \exists x\,[\,\exists y\, F_m(y,x) \;\&\; Q(x)\,] \\ \vee\;& \exists x\,[\,\neg P(x) \;\&\; \neg Q(x)\,] \end{aligned}$$

$\underline{\mathrm{Val}}^{\Sigma^0_1}(\ulcorner G_{m,n}\urcorner)$ reads then: the r.e. sets $S_1 :\equiv \{\, q \mid \exists p\, F_n(p,q) \,\}$ and $S_2 :\equiv \{\, q \mid \exists p\, F_m(p,q) \,\}$ are (weakly) inseparable by any couple of decidable r.e. sets.

3. WEAK INCOMPLETENESS OF L_1 (*under* RED)

3.1. PROPOSITION. *The species* $S := \{ \langle m,n\rangle \mid \underline{\mathrm{Val}}^{\Sigma_1^0}(\ulcorner G_{m,n}^{\omega}\urcorner) \}$ *is not* r.e..

PROOF. Assume that S is r.e.,

$$\langle m,n\rangle \in S \equiv \exists q\, S^0(q,m,n),\ S^0 \text{ prim. rec., say.}$$

Let

$$A^+ :\equiv A + \{ \underline{\mathrm{Val}}^{\Sigma_1^0}(\ulcorner G_{m,n}^{\omega}\urcorner) \mid \langle m,n\rangle \in S \}.$$

I.e.,

(1) $$\underline{\mathrm{Prov}}_{A^+}(p,\ulcorner F\urcorner) :\equiv \underline{\mathrm{Prov}}_A((p)_0, i(p,\ulcorner F\urcorner)) \ \&\ S^*(p)$$

where i is a prim. rec. function which satisfies

$$i(\langle p_0,\langle q_1,m_1,n_1\rangle,\ldots,\langle q_r,m_r,n_r\rangle\rangle,\ulcorner F\urcorner) = \ulcorner \bigwedge_{1\le j\le r} \underline{\mathrm{Val}}^{\Sigma_1^0}(\ulcorner G_{m_j,n_j}^{\omega}\urcorner) \to F\urcorner$$

and

$$S^*(p) :\equiv \forall i_{1\le i\le \underline{\mathrm{lth}}(p)}\, S^0((p)_{i,0},(p)_{i,1},(p)_{i,2}).$$

$\underline{\mathrm{Prov}}_{A^+}$ is a prim. rec. proof predicate for A^+, proven in A to satisfy the elementary derivability conditions.

Fix now a couple of r.e. sets R_1,R_2 which are proved in A to be disjoint and recursively inseparable. (Note that the proof given by ROGERS [67] p.94 thm. XII(c) for the existence of such a couple holds intuitionistically.)

(2) $$R_i \equiv \{ j \mid \exists n\, R_i^0(n,j) \} \qquad i = 1,2$$

Let

(3) $$Q_i :\equiv \{ j \mid \exists n\, [\, R_i^0(n,j) \ \&\ \forall m{<}\underline{\max}[j,n]\ \neg\underline{\mathrm{Prov}}_{A^+}(m,\ulcorner\bot\urcorner)\,] \}$$

$$\equiv: \{ j \mid \exists n\, F_{m_i}(n,j) \} \qquad (i=1,2)$$

If

(4) $$\underline{\mathrm{Prov}}_{A^+}(k,\ulcorner\bot\urcorner) \ \&\ \forall m{<}k\ \neg\underline{\mathrm{Prov}}_{A^+}(m,\ulcorner\bot\urcorner)$$

then Q_1,Q_2 are finite sets and furthermore

(5) $Q_i(j) \;:\equiv\; j \in Q_i \;(i=1,2)$ and $U_{\substack{1\\2}}(j) \;:\equiv\; \neg Q_{\substack{2\\1}}(j)$

are all recursive. Since R_1, R_2 are provably disjoint, we have also

$$\neg\exists n\,[\,\neg U_1(n) \;\&\; \neg U_2(n)\,].$$

So we have obtained

$$\forall n\,[\,U_1 n \;\vee\; \neg U_1 n\,] \;\&\; \forall n\,[\,U_2 n \;\vee\; \neg U_2 n\,] \;\&\; \neg\exists n\,[\,Q_1(n) \;\&\; U_2(n)\,] \;\&$$

$$\neg\exists n\,[\,Q_2(n) \;\&\; U_1(n)\,] \;\&\; \neg\exists n\,[\,\neg U_1(n) \;\&\; \neg U_2(n)\,]$$

i.e. $-\neg G^{\omega}_{m_1,m_2}[U_1,U_2]$ and so $\neg\underline{\mathrm{Val}}^{\Sigma^0_1}(\ulcorner G^{\omega}_{m_1,m_2}\urcorner)$. All this argument is clearly in A, i.e. -

(6) $$\exists p\,\underline{\mathrm{Prov}}_{A^+}(p,\ulcorner\bot\urcorner) \;\vdash_A\; \neg\underline{\mathrm{Val}}^{\Sigma^0_1}(\ulcorner G^{\omega}_{m_1,m_2}\urcorner).$$

But (in L_2) A is known to be sound, so by definition (1) of $\underline{\mathrm{Prov}}_{A^+}$

(7) $$\neg\exists p\,\underline{\mathrm{Prov}}_{A^+}(p,\ulcorner\bot\urcorner).$$

This implies that $Q_i = R_i$ $(i=1,2)$, and since R_1, R_2 are known to be rec. inseparable, we get by the definition of G

(8) $$\underline{\mathrm{Val}}^{\Sigma^0_1}(\ulcorner G^{\omega}_{m_1,m_2}\urcorner)$$

and so

(9) $$\vdash_{A^+}\; \underline{\mathrm{Val}}^{\Sigma^0_1}(\ulcorner G^{\omega}_{m_1,m_2}\urcorner)$$ by the definition of A^+.

Combined with (6) this gives

$$\vdash_{A^+}\; \neg\exists p\,\underline{\mathrm{Prov}}_{A^+}(p,\ulcorner\bot\urcorner)$$

which contradicts Gödel's second incompleteness theorem. □

Combining 2.6 and 3.1 we get

<u>THEOREM I</u> (weak incompleteness)

$$\vdash_{L_2+\mathrm{RED}} \neg\forall n\ [\ \underline{\mathrm{Val}}(n) \rightarrow \underline{\mathrm{Pr}}_{L_1}(n)\ ] \quad \square$$

3.2. The result of 3.1 can be classically improved by the following

<u>PROPOSITION</u>: $S := \{ \langle m,n\rangle \mid W_m, W_n$ rec. inseparable $\}$ *is* Π^0_2*-complete.*

<u>PROOF</u>. (essentially due to C. JOCKUSCH) Fix a couple R_1, R_2 of recursively inseparable r.e. sets. Let $k(e)$, $h_1(e)$, $h_2(e)$ be prim. rec. functions defined (through the s-m-n theorem) by

$$x \in W_{k(e)} \equiv \exists y{<}x\ y \in W_e$$

$$W_{h_1(e)} = W_{k(e)} \cap R_1\ ; \qquad W_{h_2(e)} = W_{k(e)} \cap R_2$$

Then: (i) W_e finite $\Rightarrow$ $W_{k(e)}$ finite

$\Rightarrow$ $W_{h_1(e)}$ and $W_{h_2(e)}$ finite

$\Rightarrow$ $\langle h_1(e), h_2(e)\rangle \notin S$

(ii) W_e infinite $\Rightarrow$ $W_{k(e)} = N$

$\Rightarrow$ $W_{h_i(e)} = R_i$ $\quad (i=1,2)$

$\Rightarrow$ $\langle h_1(e), h_2(e)\rangle \in S$

So the set

$$I := \{ e \mid W_e \textit{ is infinite} \}$$

reduces to S. But I is known to be Π^0_2-complete (cf. ROGERS [67] p.326, 1.3), and hence S is also Π^0_2-complete. $\square$

By a double-negation translation of the proposition, we get that the following is provable in intuitionistic arithmetic *A*.

(1) $\forall s \{ P(s) \leftrightarrow \neg\neg\exists e,r \forall n [n \in s \leftrightarrow T(e,n,r) \ \& \ Ur \in s_0] \}$

where P is a p.r. predicate stating that s is a code of a unary predicate of the form

(2) $\forall x \neg\neg\exists y \, Q(x,y,n)$, Q q.f.,

$n \in s$ is (2), and s_0 is a specific predicate of the from (2) which states that $W_{(n)_0}$ and $W_{(n)_1}$ are weakly recursively inseparable.

One can now prove theorem I as a corallary of (1) above in a straightforward manner.

3.3. A SPECIFIC EXAMPLE TO THE INCOMPLETENESS OF L_1

PROPOSITION. *There are* m_1, m_2 *s.t.*

$$\vdash_{L_2+\mathrm{RED}} \underline{\mathrm{Val}}(\ulcorner A_{m_2} \to G_{m_1,m_2} \urcorner) \ \& \ \neg\underline{\mathrm{Pr}}_{L_1}(\ulcorner A_{m_2} \to G_{m_1,m_2} \urcorner)$$

FIRST PROOF. Let R_1, R_2 be as in 3.1, $R_i = W_{m_i}$ (i=1,2). As in 3.1 (8) we then have

$$\vdash_{L_2} \underline{\mathrm{Val}}^{\Sigma_1^0}(\ulcorner G^{\omega}_{m_1,m_2} \urcorner)$$

which by 2.6 implies

(1) $$\vdash_{L_2+\mathrm{RED}} \underline{\mathrm{Val}}(\ulcorner A_{m_2} \to G_{m_1,m_2} \urcorner)$$

But $A_{m_2} \to G_{m_1,m_2}$ is obviously not valid classically, and so (by completeness of L_1^C relative to classical validity)

$$\nvdash_{L_1^C} A_{m_2} \to G_{m_1,m_2}$$

and then of course

$$\nvdash_{L_1} A_{m_2} \to G_{m_1,m_2} \qquad \square$$

<u>SECOND PROOF</u>. Let R_1, R_2 be as above, and let Q_1, Q_2, m_1, m_2 be defined as in 3.1, but with $\underline{\text{Prov}}_{A_0}$ in place of $\underline{\text{Prov}}_{A^+}$, where A_0 is p.r. (i.e. - qunatifier free) arithmetic.

Since

$$\vdash_{L_2} \neg\exists n\, \underline{\text{Prov}}_{A_0}(n, \ulcorner\bot\urcorner)$$

we conclude (1) as in the first proof.

On the other hand, if

(2) $$\vdash_{L_1} A_{m_2} \rightarrow G_{m_1,m_2}[P,Q]$$

then for every binary arithmetical predicates U_1, U_2

(3) $$\vdash_{A_0} \forall k\, G^{\omega}_{m_1,m_2}[\lambda p.U_1(k,p), \lambda p.U_2(k,p)].$$

Let, in particular

$$\underset{2}{U_1}(k,p) \;:\equiv\; \neg\exists m\leq k\; \underset{1}{R_2^0}(m,p).$$

$\lambda p.U_i(k,p)$ is the U_i of 3.1 (5) if 3.1 (4) holds for k. Thus, as in 3.1

(4) $$\vdash_{A_0} \underline{\text{Prov}}_{A_0}(k, \ulcorner\bot\urcorner) \;\&\; \forall m<k\, \neg\underline{\text{Prov}}_{A_0}(m, \ulcorner\bot\urcorner)$$
$$\rightarrow \neg G_{m_1,m_2}[\lambda p.U_1(k,p), \lambda p.U_2(k,p)].$$

Combining now (3) and (4) we have

$$\vdash_{A_0} \neg\exists k\, \underline{\text{Prov}}_{A_0}(k, \ulcorner\bot\urcorner)$$

contradicting Gödel's second incompleteness theorem. □

3.4. INCOMPLETENESS W.R.T. VALIDITY WITH CHOICE PARAMETERS

Let α be a variable for choice sequences, and assume that for the kind of choice sequences considered we have

(1) $$\forall\alpha\neg\neg\exists e\ \alpha \simeq \{e\}$$

which is the case for the choice sequences investigated by KREISEL-TROELSTRA [70] (cf. 6.2.1 there).

(1) implies quite trivially for every negated formula $\neg A(\alpha)$

(2) $$\forall e\ \neg A(\{e\}) \rightarrow \forall\alpha\ \neg A(\alpha).$$

Let us refer now to a notion of validity with choice parameters, $\underline{Val}^{CS}$ say; i.e. -

$$\underline{Val}^{CS}(\ulcorner G \urcorner) \quad :\equiv \quad \forall\alpha\ \underline{Val}(\ulcorner G^{\alpha} \urcorner)$$

where G^{α} comes from G by replacing each atomic subformula $P(t_1,\ldots,t_n)$ by $P(\alpha,t_1,\ldots,t_n)$. (This is not weaker than allowing the choice parameters to be distinct for each predicate letter, as can readily be seen.)

A straightforward observation shows now that the whole treatment of sec. 2, 3.1, 3.2 and 3.3 (second proof) works when $\underline{Val}$ is replaced by $\underline{Val}^{CS}$ (but $\underline{Val}^{\Sigma^0_1}$ remains unchanged), provided of course the schema RED is generalized to

(3) $$\mathrm{RED}^{CS} : \forall\alpha\ \{ \forall x\ [\ P(\alpha,x) \vee \neg P(\alpha,x)\ ] \rightarrow \neg\neg\exists e \forall n\ [\ P(\alpha,n) \leftrightarrow \{e\}n \simeq 0\ ]\}$$

But by (2) RED^{CS} is implied outright by RED, since RED is negative, and (3) with α varying over total recursive functions is just a special case of (the quantified variant of) RED.

4. L_1 IS NOT Σ_1^0-ABSOLUTE FOR A

4.1. DEFINITION. Let C be a class of number theoretic predicates. A sentence $G[P_1,\dots,P_k]$ of L_1 is said to be *C-absolute for S* iff

$$\vdash_S \; G[P_1^*,\dots,P_k^*]$$

for every $P_1^*,\dots,P_k^*$ in C. We thence define more formally Σ_1^0-absoluteness for A as

$$(1) \quad \underline{\mathrm{Abs}}_A^{\Sigma_1^0}(\ulcorner G[P_1,\dots,P_k]\urcorner) \;:\equiv\; \forall e_1,\dots,e_k \exists p\; \underline{\mathrm{Prov}}_A(p,\ulcorner G[W_{e_1},\dots,W_{e_k}]\urcorner)$$

for every schema G of L_1 (whose predicate letters are among $P_1,\dots,P_k$) where, if $P_j \equiv P_j^n$,

$$W_{e_j}(m_1,\dots,m_n) \;:\equiv\; \{e_j\}\langle m_1,\dots,m_n\rangle \simeq 0$$

L_1 is said to be *C-absolute for S* if

$$L_1 \;=\; \{\, G \mid G \textit{ is C-absolute for } S \,\}.$$

In LEIVANT [75] L_1 is shown to be Π_2^0-absolute for A. Here we show that L_1 is not Σ_1^0-absolute (even for a weak fragment of A).

4.2. Fix a schema $G \equiv G[Eq,Z,S,F_1,\dots,F_n;\, P_1,\dots,P_k]$ and write

$$\begin{aligned} G^e \;&:\equiv\; G[W_{(e)_0},\dots,W_{(e)_{n+k+2}}] \\ &\equiv:\; G[Eq^*,\dots,P_k^*] \end{aligned}$$

The Σ_1^0 predicates Z^* and S^* define a structure isomorphic to $\langle\omega,0,\lambda x.x^+\rangle$ through a general recursive function ν, as follows.

$$\nu(0) := \mu n.Z^*(n)$$

$$\nu(m+1) := \mu n.S^*(\nu(m),n)$$

$$N_e(m) :\equiv \exists n\ \nu(n) \simeq m.$$

N_e is uniquely determined by e, and we define the arithmetical formula G^{e,N_e} by restricitng all quantifiers of G in G^e to N_e.

PROPOSITION

(1) $$\vdash_A \underline{\mathrm{Abs}}_A^{\Sigma_1^0}(\ulcorner G^\omega \urcorner) \rightarrow \forall e\ \underline{\mathrm{Pr}}_A(\ulcorner A_n^e \rightarrow G^{e,N_e}\urcorner)$$

for every formula G *of* L_1*, where* n *is a bound on the indices of the predicates* F_i *which occur in* G.

PROOF. Let A be given by a Gentzen natural deduction system (cf., e.g., PRAWITZ [71]). Fix $G \equiv G[Eq,\ldots,F_n;\ P_1,\ldots,P_k]$ as above, and assume the premise of (1),

(2) $$\forall e\ \underline{\mathrm{Pr}}_A(\ulcorner (G^\omega)^e \urcorner)$$

Let e be given, $Eq^* :\equiv W_{(e)_0}$, $Z^* :\equiv W_{(e)_1}, \ldots;\ N :\equiv N_e$. For a formula H of A write H^N for the formula which comes from H by interpreting $=, 0, \lambda x.x^+, \lambda\vec{x},y.f_1(\vec{x}) = y, \ldots$ by Eq^*, Z^*, S^*, $F_1^*, \ldots$, and restricting quantifiers to N. It is easily verified (in A) that for each inference rule

$$\frac{\langle J_i \rangle_i}{K}\ \rho$$

of A, the implication

$$\bigwedge_i J_i^N \rightarrow K^N$$

is derivable in $A + A_m[Eq^*,\ldots,F_m^*] \equiv A + A_m^e$ where m is a bound on the indices of p.r. functions occurring in $\{J_i\}_i, K$. (Note that this would have failed for the induction rule if the entire domain of the successor relation S^* was taken as N_e.)

Let now

$$e' := \langle (e)_{n+2},\dots,(e)_{n+k+2}\rangle$$

so

$$(G^{\omega})^{e'} \equiv G[=,0,\dots;\ W_{(e')_0},\dots,W_{(e')_k}].$$

By (2) there is a derivation Δ for

$$\vdash_{A} (G^{\omega})^{e'}$$

Since adding prim. rec. functions to a system $A^* \supseteq A$ is conservative over A^*, we may assume that no p.r. function with index > n occur in Δ. By the discussion above, using induction on the length of Δ we therefore may conclude that

$$((G^{\omega})^{e'})^{N} \equiv G^{e,N_e}$$

is derivable in $A + A_n^e$, as required. □

4.3. Let now $E_{m,n} \equiv E[F_m,F_n;\ P,Q]$ be as in 2.6.

<u>PROPOSITION</u>

$$\vdash_{A} \underline{\mathrm{Abs}}_{A}^{\Sigma_1^0}(\ulcorner E_{m,n}{}^{\omega}\urcorner) \rightarrow \underline{\mathrm{Abs}}_{A}^{\Sigma_1^0}(\ulcorner A_n \rightarrow E_{m,n}\urcorner)$$

<u>PROOF</u>. Assume

(1) $\qquad \forall e\ \underline{\mathrm{Pr}}_{A}(\ulcorner (E_{m,n}{}^{\omega})^{e}\urcorner)$

and fix an e,

$$Eq^* :\equiv W_{(e)_0},\ \dots\ ,\ F_n^* :\equiv W_{(e)_{n+2}};$$

$$P^* :\equiv W_{(e)_{n+3}}, \qquad Q^* :\equiv W_{(e)_{n+4}}.$$

By 4.2 now (1) implies

(2) $\qquad \vdash_{A} A_n^e \rightarrow E_{m,n}^{e,N_e}$

but since $E_{m,n}$ is existential,

$$\vdash_{A} \; E_{m,n}^{e,N_e} \rightarrow E_{m,n}^{e} .$$

So (2) yields

$$\vdash_{A} \; (A_n \rightarrow E_{m,n})^e$$

which quantifying over e reads $\underline{\mathrm{Abs}}_{A}^{\Sigma_1^0}(\ulcorner A_n \rightarrow E_{m,n}\urcorner)$. □

4.4. THEOREM II. There are m_1, m_2 s.t.

$$\vdash_{A} \; \underline{\mathrm{Abs}}_{A}^{\Sigma_1^0}(\ulcorner A_{m_2} \rightarrow E_{m_1,m_2}\urcorner) \;\&\; \neg\underline{\mathrm{Pr}}_{L_1}(\ulcorner A_{m_2} \rightarrow E_{m_1,m_2}\urcorner).$$

PROOF. Let $R_1, R_2, Q_1, Q_2, m_1, m_2$ be as in the second proof of 3.3. R_1, R_2 are A-proven to be rec. inseparable, and since $\vdash_{A} \neg\exists x\, \underline{\mathrm{Prov}}_{A_0}(x, \ulcorner\bot\urcorner)$,

$$\vdash_{A} \; \text{" } Q_1 \equiv R_1 \;\&\; Q_2 \equiv R_2 \text{ "}$$

and so

$$\vdash_{A} \; \forall e_1, e_2 \, E_{m_1,m_2}^{\omega}[W_{e_1}, W_{e_2}]$$

which implies

$$\vdash_{A} \; \underline{\mathrm{Abs}}_{A}^{\Sigma_1^0}(\ulcorner E_{m_1,m_2}^{\omega}\urcorner).$$

By 4.3 then

(1) $$\vdash_{A} \; \underline{\mathrm{Abs}}_{A}^{\Sigma_1^0}(\ulcorner A_{m_2} \rightarrow E_{m_1,m_2}\urcorner).$$

On the other hand we have from 3.3

(2) $$\vdash_{A} \; \neg\underline{\mathrm{Pr}}_{L_1}(\ulcorner A_{m_2} \rightarrow E_{m_1,m_2}\urcorner).$$ □

REFERENCES

DALEN, D. VAN [72]: Lectures on Intuitionism; in *Cambridge Summer School in Mathematical Logic* (eds. Mathias and Rogers) (Spinger's Lecture Notes vol. 337), pp.1-94.

KREISEL, G. [62]: On weak completeness of intuitionistic predicate logic; *JSL* 27 (1962), pp.139-158.

KREISEL, G. [70]: Church's thesis: a kind of reducibility axiom of constructive mathematics; in *Intuitionism and Proof Theory* (eds. Kino, Myhill, Vesley) (North Holland, Amsterdam, 1970), pp.121-150.

LEIVANT, D. [72]: Notes on completeness of the intuitionistic predicate calculus, *Report ZN* 40/72 (Mathematisch Centrum, Amsterdam, 1972).

LEIVANT, D. [75]: Absoluteness of intuitionistic logic; *Report ZW* 45/75 (Mathematisch Centrum, Amsterdam, 1975).

LIFSHITS, V. [74]: Privately circulated memo, 1974.

PRAWITZ, D. [65]: *Natural Deduction* (Almqvist & Wiksell, Stockholm, 1965).

PRAWITZ, D. [71]: Ideas and results of Proof Theory; in *Proceedings of the Second Scandinavian Logic Symposium* (ed. Fenstand) (North Holland, Amsterdam 1971), pp.235-307.

ROGERS, H. [67]: *The Theory of Recursive Functions and Effective Computability* (McGraw-Hill, New York, 1967).

TROELSTRA, A.S. [74]: Markov's principle and Markov's rule for theories of choice sequences, *Report* 74-12, Dept. of Mathematics, University of Amsterdam. To appear in the proceedings of the A.S.L. meeting at Kiel.

VAUGHT, R.E. [60]: Sentences true in all constructive models, *JSL* 25 (1960), pp.39-53.